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Lagrangian Probability Distributions of Turbulent Flows

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Abstract

We outline a statistical theory of turbulence based on the Lagrangian formulation of fluid motion. We derive a hierarchy of evolution equations for Lagrangian N-point probability distributions as well as a functional equation for a suitably defined probability functional which is the analog of Hopf's functional equation. Furthermore, we adress the derivation of a generalized Fokker-Plank equation for the joint velocity-position probability density of N fluid particles.

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The strategy of approaching the phenomenon of fully developed turbulence by considering the statistics of Lagrangian fluid particles has a long tradition dating back to the early works of Taylor [1], Richardson [2] (for an overview, see [3], [4], [5]). Recently, interest in the Lagrangian statistics has been renewed by rigorous results on passive scalar advection in the Kraichnan model [6]. Additionally, experimental progress has opened the way to gain accurate data on the motion of tracer particles which allows one to directly evaluate the statistics of the particle acceleration [7], [8] ,[9], [10]. These experiments will also shed light on the physics of relative dispersion of fluid particles [11].

A statistical formulation of the problem of turbulence starting from the Lagrangian point of view will lead to considerable progress in modeling turbulent flows by Lagrangian pdf method [12], [13] [15]. This method can successfully deal with passive scalar transport, turbulent flows involving chemical reactions or combustion [14]. In this approach, which origionally dates back to Oboukhov [16], Fokker-Planck equations are used to model the statistical behaviour of fluid particles. Although this is an appealing approach, only few efforts have been made to relate the Lagrangian turbulence statistics to diffusion processes by a direct investigation of the Navier-Stokes equation. An exception is the work of Heppe [21], who uses a projector-formalism to obtain a generalized diffusion equation for the joint velocity-position probability distribution of one particle.

In the following we shall present a hierarchy of evolution equations for N-point probability distributions describing the behaviour of N particles in a turbulent flow. The hierarchy is in close analogy to the one presented by Lundgren [23] and Ulinich and Ljubimov [24] for the probability distributions of the Eulerian velocity field (see [4]). Additionally we derive a functional equation for a suitably defined probability functional, which is the analog of Hopf's functional equation [22] and, therefore, is a concise formulation of the problem of turbulence in the Lagrangian formulation.

Then we shall adress the question whether a type of diffusion process in the sense of the random force method advocated by Novikov [17] can approximate the motion of N particles. We shall derive a generalized Fokker-Planck equation involving memory terms, which determines the evolution of the N-point probability distribution. However, the drift and diffusion terms of the generalized Fokker-Planck equation are expressed in terms of conditional probabilities of higher order such that the problem remains unclosed. In a following paper we shall adress closure approximations.

I. FORMAL LAGRANGIAN DESCRIPTION

In the present section we shall introduce a formal Lagrangian description of fluid flow, which will be suitable for formulating evolution equations for statistical quantities. We consider the Navier-Stokes equation for an incompressible Eulerian velocity field $\mathbf{u}(\mathbf{x},t)$:

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)=-\nabla p(\mathbf{x},t)+\nu\Delta\mathbf{u}(\mathbf{x},t) \qquad . \tag{1}$$

In order to obtain a closed evolution equation one has to express the pressure as a functional of the velocity field. As is well known the pressure is governed by the Poisson equation

$$\Delta p(\mathbf{x},t)=-\nabla\cdot[\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)] \qquad . \tag{2}$$

In the case of a finite fluid volume V appropriate boundary conditions have to be formulated. Let us consider the case with a prescribed normal component of the pressure gradient. The solution of this von Neumann boundary value problem reads

$$\begin{aligned} p(\mathbf{x},t) &= \frac{1}{4\pi}\int_V d\mathbf{x}G(|\mathbf{x}-\mathbf{x}'|)\nabla\cdot[\mathbf{u}(\mathbf{x}',t)\cdot\nabla\mathbf{u}(\mathbf{x}',t)] \\ &+ \frac{1}{4\pi}\int_{\delta V} G(\mathbf{x}-\mathbf{x}')\nabla p(\mathbf{x}',t)\cdot\mathbf{dA} \end{aligned} \tag{3}$$

Here, $G(\mathbf{x}-\mathbf{x}')$ denotes the Green's function

$$\begin{aligned} \Delta G(\mathbf{x}-\mathbf{x}') &= -4\pi\delta(\mathbf{x}-\mathbf{x}') \\ \mathbf{n}\cdot\nabla_{\mathbf{x}'}G(\mathbf{x}-\mathbf{x}') &= -\frac{4\pi}{S} \qquad \mathbf{x}'\in\delta V \qquad . \end{aligned} \tag{4}$$

S is the area of the surface δV enclosing the fluid. The boundary condition is incorporated into the definition of the Green's function.

In the case of an infinitely extended fluid volume the boundary term vanishes and the Green's function is given by

$$G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|} \qquad . \tag{5}$$

Now we can state the Navier-Stokes equation:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) &+ \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) \\ &= -\int_V d\mathbf{x}'\Gamma(\mathbf{x},\mathbf{x}'):\mathbf{u}(\mathbf{x}',t):\mathbf{u}(\mathbf{x}',t)+\mathbf{F}(\mathbf{x},t) \\ &+ \nu\int d\mathbf{x}'L(\mathbf{x},\mathbf{x}')\mathbf{u}(\mathbf{x}',t) \qquad . \end{aligned} \tag{6}$$

In order to obtain a convenient formulation for the subsequent statistical treatment we have introduced the following notation for the pressure and the viscous terms, respectively:

$$
\begin{aligned}
[\Gamma(\mathbf{x},\mathbf{x}'):\mathbf{u}(\mathbf{x}',t):\mathbf{u}(\mathbf{x}',t)]_\alpha &= \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma}\frac{1}{4\pi}G(\mathbf{x}-\mathbf{x}')u_\beta(\mathbf{x}',t)u_\gamma(\mathbf{x}',t) \\
\mathbf{F}(\mathbf{x},t) &= -\frac{1}{4\pi}\int_{\delta V}\nabla_\mathbf{x} G(\mathbf{x}-\mathbf{x}')\mathbf{n}\cdot\nabla p(\mathbf{x}',t)]\cdot \mathbf{dA} \\
L(\mathbf{x},\mathbf{x}') &= \Delta_\mathbf{x}\delta(\mathbf{x}-\mathbf{x}') \qquad .
\end{aligned} \tag{7}
$$

The quantity $L(\mathbf{x},\mathbf{x}')$ is a generalized function and is defined in a formal sense. The term $\mathbf{F}(\mathbf{x},t)$ is due to boundary conditions, c.f. eq. (3). Note, that we have assumed that the normal pressure gradient is prescribed at the boundary.

Now we turn to a Lagrangian formulation of the equation of fluid motion. To this end we consider a Lagrangian path $\mathbf{X}(t,\mathbf{y})$ of a fluid particle, which initially was located at $\mathbf{X}(t_0,\mathbf{y})=\mathbf{y}$. The velocity of the particle is given in terms of the Eulerian velocity field $\mathbf{u}(\mathbf{X}(t,\mathbf{y}),t)$, whereas the Navier-Stokes equation takes the form

$$
\begin{aligned}
\frac{d}{dt}\mathbf{X}(t,\mathbf{y}) &= \mathbf{u}(\mathbf{X}(t,\mathbf{y}),t) \qquad (8)\\
\frac{d}{dt}\mathbf{u}(\mathbf{X}(t,\mathbf{y}),t) &= -\int d\mathbf{x}'\Gamma(\mathbf{X}(t,\mathbf{y}),\mathbf{x}'):\mathbf{u}(\mathbf{x}',t):\mathbf{u}(\mathbf{x}',t) \\
&+ F(\mathbf{X}(t,\mathbf{y}),t)+\nu\int d\mathbf{x}'L(\mathbf{X}(t,\mathbf{y}),\mathbf{x}')\mathbf{u}(\mathbf{x}',t) \qquad .
\end{aligned}
$$

For the evaluation of the integrals we perform a coordinate transformation

$$
\mathbf{x}'=\mathbf{X}(t,\mathbf{y}') \qquad . \tag{9}
$$

Due to incompressibility, the Jacobian equals unity:

$$
Det[\frac{\partial \mathbf{X}_\alpha(t,\mathbf{y})}{\partial y_\beta}]=1 \qquad . \tag{10}
$$

Now we define the Lagrangian velocity $\mathbf{U}(t,\mathbf{y})$ according to

$$
\mathbf{U}(t,\mathbf{y})=\mathbf{u}(\mathbf{X}(t,\mathbf{y}),t) \qquad . \tag{11}
$$

As a result we end up with the following Lagrangian formulation of the basic fluid dynamics equation:

$$
\begin{aligned}
\frac{d}{dt}\mathbf{X}(t,\mathbf{y}) &= \mathbf{U}(t,\mathbf{y}) \qquad (12)\\
\frac{d}{dt}\mathbf{U}(t,\mathbf{y})) &= -\int d\mathbf{y}'\Gamma[\mathbf{X}(t,\mathbf{y}),\mathbf{X}(t,\mathbf{y}')]:\mathbf{U}(t,\mathbf{y}'):\mathbf{U}(t,\mathbf{y}') \\
&+ F(\mathbf{X}(t,\mathbf{y}),t)+\nu\int d\mathbf{y}'L[\mathbf{X}(t,\mathbf{y}),\mathbf{X}(t,\mathbf{y}')]\mathbf{U}(t,\mathbf{y}') \qquad .
\end{aligned}
$$

We have obtained a representation of the acceleration of a Lagrangian particle, although this expression is formal due to the appearance of the generalized function $L(\mathbf{x}, \mathbf{x}')$. Nevertheless, we shall find that one recovers meaningful and well defined expressions when one proceeds to a statistical formulation.

II. EULERIAN AND LAGRANGIAN PROBABILITY DISTRIBUTIONS

The purpose of the present section is to develop a statistical description of the fluid motion. The central quantities will be N-point position-velocity probability densities and, as the most general quantity, a velocity-position probability functional.

We start by defining the N-point Lagrangian velocity-position distribution function

$$
\begin{aligned}
&f^N(\{\mathbf{u}_j, \mathbf{x}_j, \mathbf{y}_j\}; t) = \\
&\quad < \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)]\delta[\mathbf{u}_1 - \mathbf{U}(t, \mathbf{y}_1)] .... \delta[\mathbf{x}_N - \mathbf{X}(t, \mathbf{y}_N)]\delta[\mathbf{u}_N - \mathbf{U}(t, \mathbf{y}_N)] > \qquad . \qquad (13)
\end{aligned}
$$

This distribution function allows one to statistically characterize the behaviour of a swarm of N fluid particles. The brackets indicate averaging with respect to a suitably defined statistical ensemble. Since we have $\mathbf{X}(t = 0, \mathbf{y}) = \mathbf{y}$ the initial condition at $t = 0$ reads:

$$
\begin{aligned}
&f^N(\{\mathbf{u}_j, \mathbf{x}_j, \mathbf{y}_j\}; t = 0) = \\
&\quad \delta(\mathbf{x}_1 - \mathbf{y}_1) .... \delta(\mathbf{x}_N - \mathbf{y}_N) g^N(\{\mathbf{u}_j, \mathbf{y}_j\}) \qquad , \qquad (14)
\end{aligned}
$$

where $g^N(\{\mathbf{u}_j, \mathbf{y}_j\})$ is the joint probability distribution for the N velocities of the fluid particles at initial time $t = 0$. In the stationary case, $g^N(\{\mathbf{u}_j, \mathbf{y}_j\})$ is the Eulerian probability distribution defined below.

The N-point probability distribution for the particle locations, $p^N(\{\mathbf{x}_j, \mathbf{y}_j\}; t)$ is obtained by integration with respect to the velocities $\mathbf{u}_j$ and is the central quantity in the theory of dispersion of particles in turbulence. Alternatively, we may integrate eq. (13) over the spatial variables and obtain the pure velocity probability distributions of the particles.

It is convenient to additionally define the corresponding Eulerian probability distribution

$$
f_E^N(\mathbf{u}_1, \mathbf{x}_1; ..; \mathbf{u}_N, \mathbf{x}_N; t) =< \delta(\mathbf{u}_1 - \mathbf{u}(\mathbf{x}_1, t)) .. \delta(\mathbf{u}_N - \mathbf{u}(\mathbf{x}_N, t)) > \qquad . \qquad (15)
$$

These probability distributions are obtained from the corresponding Lagrangian (13) ones by integration with respect to $\mathbf{y}_j$, since the fluid flow is incompressible such that the Jacobian

(10) equals unity. Furthermore, the Eulerian probability distributions have to fullfill the following consistency conditions

$$\nabla_{\mathbf{x}_i} f_E^N(\mathbf{u}_1,\mathbf{x}_1;..;\mathbf{u}_N,\mathbf{x}_N;t) = \tag{16}$$
$$[-\nabla_{\mathbf{x}'}\nabla_{\mathbf{u}_i}\cdot\int d\mathbf{u}'\mathbf{u}' f_E^{N+1}(\mathbf{u}',\mathbf{x}',\mathbf{u}_1,\mathbf{x}_1;..;\mathbf{u}_N,\mathbf{x}_N;t)]_{\mathbf{x}'=\mathbf{x}_i} \qquad i=1,..,N \qquad .$$

These relationships are a consequence of the fact that the spatial dependency of the probability distribution enters via the velocity field $\mathbf{u}(\mathbf{x},t)$.

Finally, we mention that it is also straightforward to define mixed Eulerian and Lagrangian probability distributions. They can be obtained from the Lagrangian probability distribution by integration over the variables $\mathbf{y}_i$, for which an Eulerian description is performed:

$$\begin{aligned}
& f_{E|L}^N(\mathbf{u}_1,\mathbf{x}_1;..;\mathbf{u}_m,\mathbf{x}_m;\mathbf{u}_{m+1},\mathbf{x}_{m+1},\mathbf{y}_{m+1};..;\mathbf{u}_n,\mathbf{x}_n,\mathbf{y}_n;t) = \\
& \quad < \delta(\mathbf{u}_1-\mathbf{u}(\mathbf{x}_1,t))..\delta(\mathbf{u}_m-\mathbf{u}(\mathbf{x}_m,t)) \\
& \quad \delta[\mathbf{x}_{m+1}-\mathbf{X}(t,\mathbf{y}_{m+1})]\delta[\mathbf{u}_{m+1}-\mathbf{u}(\mathbf{X}(t,\mathbf{y}_{m+1}),t)].... \\
& \quad \delta[\mathbf{x}_n-\mathbf{X}(t,\mathbf{y}_n)]\delta[\mathbf{u}_n-\mathbf{u}(\mathbf{X}(t,\mathbf{y}_n),t)] > \\
& \quad = \int d\mathbf{y}_1...d\mathbf{y}_m f_L^N(\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1;..;\mathbf{u}_n,\mathbf{x}_n,\mathbf{y}_n;t) \qquad .
\end{aligned} \tag{17}$$

Since the Navier-Stokes equation form a classical field theory a complete description of the Lagrangian turbulence statistics is contained in the probability density functional $P[\mathbf{x}(\mathbf{y}),\mathbf{u}(\mathbf{y}),t]$ defined by an average over functional delta distributions:

$$P[\mathbf{x}(\mathbf{y}),\mathbf{u}(\mathbf{y}),t] =< D[\mathbf{x}(\mathbf{y})-\mathbf{X}(t,\mathbf{y})]D[\mathbf{u}(\mathbf{y})-\mathbf{U}(t,\mathbf{y})] > \qquad . \tag{18}$$

A probability functional for the field $\mathbf{X}(t,\mathbf{y}')$ is obtained by functional integration with respect to the velocity:

$$G[\mathbf{x}(\mathbf{y}),t] = \int D\mathbf{u}(\mathbf{y})P[\mathbf{x}(\mathbf{y}),\mathbf{u}(\mathbf{y}),t] \qquad . \tag{19}$$

The functional Fouriertransform of the quantity (18) is the characteristic functional

$$Z[\eta(\mathbf{y}),\mathbf{k}(\mathbf{y}),t] =< e^{i\int d\mathbf{y}[\eta(\mathbf{y})\cdot\mathbf{U}(t,\mathbf{y})+\mathbf{k}(\mathbf{y})\cdot\mathbf{X}(\mathbf{y},t)]} > \qquad . \tag{20}$$

This is the Lagrangian analog of the functional introduced by Hopf [22].

III. HIERARCHY OF EVOLUTION EQUATIONS FOR THE N-POINT PROBABILITY DISTRIBUTION FUNCTIONS

In the present section we shall obtain an infinite chain of evolution equations for the Lagrangian probability distribution functions $f^N(\{\mathbf{u}_j, \mathbf{x}_j, \mathbf{y}_j\}; t)$, $N = 1, 2, ...$ For the sake of simplicity we consider a fluid in an infinitely extended space. Therefore, we do not have to take into account boundary terms. The extention to the case of a fluid in a bounded area is straightforward.

We start by considering the one-point pdf $f^1(\mathbf{u}, \mathbf{x}, \mathbf{y}; t)$, for which we try to obtain an evolution equation by calculating the temporal derivative:

$$\begin{aligned}
&\frac{\partial}{\partial t} f^1(\mathbf{u}_1, \mathbf{x}_1, \mathbf{y}_1; t) = \\
&\quad \frac{\partial}{\partial t} < \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] > \\
&\quad = -\nabla_{\mathbf{x}_1} \cdot < \dot{\mathbf{X}}(t, \mathbf{y}_1) \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] > \\
&\quad -\nabla_{\mathbf{u}_1} \cdot < \dot{\mathbf{u}}(\mathbf{X}(t, \mathbf{y}_1), t) \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] > \qquad . \qquad (21)
\end{aligned}$$

For the first term we can use the relationship

$$\begin{aligned}
&< \dot{\mathbf{X}}(t, \mathbf{y}_1) \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] >= \\
&\quad < \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t) \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] >= \\
&\quad \mathbf{u}_1 < \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] > \qquad . \qquad (22)
\end{aligned}$$

In order to evaluate the second term we have to insert the Lagrangian formulation of the Navier-Stokes equation (12):

$$\begin{aligned}
&< \dot{\mathbf{u}}(\mathbf{X}(t, \mathbf{y}_1), t) \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] >= \\
&\quad - \int d\mathbf{y}' < \Gamma[\mathbf{X}(t, \mathbf{y}), \mathbf{X}(t, \mathbf{y}')] : \mathbf{u}(\mathbf{X}(t, \mathbf{y}'), t) : \mathbf{u}(\mathbf{X}(t, \mathbf{y}'), t) \\
&\quad \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] > \\
&\quad + \nu \int d\mathbf{y}' < L(\mathbf{X}(t, \mathbf{y}), \mathbf{X}(t, \mathbf{y}')) \mathbf{u}(\mathbf{X}(t, \mathbf{y}'), t) \\
&\quad \delta[\mathbf{x}_1 - \mathbf{X}(t, \mathbf{y}_1)] \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}_1), t)] > \qquad . \qquad (23)
\end{aligned}$$

The aim is to relate this term in some way to a Lagrangian probability distribution. This can be achieved by inserting the identity

$$1 = \int d\mathbf{u}_2 \int d\mathbf{x}_2 \delta[\mathbf{x}_2 - \mathbf{X}(t, \mathbf{y}')] \delta[\mathbf{u}_2 - \mathbf{u}(\mathbf{X}(t, \mathbf{y}'), t)] \qquad . \qquad (24)$$

As a result one obtains:

$$
\begin{aligned}
&< \dot{\mathbf{u}}(\mathbf{X}(t,\mathbf{y}_1),t)\delta[\mathbf{x}_1-\mathbf{X}(t,\mathbf{y}_1)]\delta[\mathbf{u}_1-\mathbf{u}(\mathbf{X}(t,\mathbf{y}_1),t)] >= \\
&-\int d\mathbf{u}_2\int d\mathbf{x}_2\int d\mathbf{y}_2\Gamma(\mathbf{x}_1,\mathbf{x}_2):\mathbf{u}_2:\mathbf{u}_2 f^2(\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1;\mathbf{u}_2,\mathbf{x}_2,\mathbf{y}_2;t) \\
&+\nu\int d\mathbf{u}_2\int d\mathbf{x}_2\int d\mathbf{y}_2 L(\mathbf{x}_1,\mathbf{x}_2)\mathbf{u}_2 f^2(\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1;\mathbf{u}_2,\mathbf{x}_2,\mathbf{y}_2;t) \qquad . \qquad (25)
\end{aligned}
$$

We now combine the above formulas to obtain the evolution equation for the Lagrangian probability distribution $f^1(\{\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1\};t)$:

$$
\begin{aligned}
&\frac{\partial}{\partial t}f^1(\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1;t)+\mathbf{u}_1\cdot\nabla_{\mathbf{x}_1}f^1(\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1;t) \qquad (26)\\
&=\nabla_{\mathbf{u}_1}\cdot\int d\mathbf{u}_2\int d\mathbf{x}_2\int d\mathbf{y}_2\Gamma(\mathbf{x}_1,\mathbf{x}_2):\mathbf{u}_2:\mathbf{u}_2 f^2(\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1;\mathbf{u}_2,\mathbf{x}_2,\mathbf{y}_2;t) \\
&-\nu\nabla_{\mathbf{u}_1}\cdot\int d\mathbf{u}_2\int d\mathbf{x}_2\int d\mathbf{y}_2 L(\mathbf{x}_1,\mathbf{x}_2)\mathbf{u}_2 f^2(\mathbf{u}_1,\mathbf{x}_1,\mathbf{y}_1;\mathbf{u}_2,\mathbf{x}_2,\mathbf{y}_2;t) \qquad .
\end{aligned}
$$

Due to the nonlocality of the pressure and the viscous term, which relates the Lagrangian path of a particle under consideration to the paths of different particles, the evolution equation of the one-particle pdf is linked to the two-particle pdf.

It is straightforward to prove that the N-point distribution function fullfills the evolution equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t)+\sum_i\mathbf{u}_i\cdot\nabla_{\mathbf{x}_i}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \qquad (27)\\
&=\sum_i\nabla_{\mathbf{u}_i}\cdot\int d\mathbf{u}\int d\mathbf{x}\int d\mathbf{y}\Gamma(\mathbf{x}_i,\mathbf{x}):\mathbf{u}:\mathbf{u}f^{N+1}(\mathbf{u},\mathbf{x},\mathbf{y};\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \\
&-\nu\sum_i\nabla_{\mathbf{u}_i}\cdot\int d\mathbf{u}\int d\mathbf{x}\int d\mathbf{y}L(\mathbf{x}_i,\mathbf{x})\mathbf{u}f^{N+1}(\mathbf{u},\mathbf{x},\mathbf{y};\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \qquad .
\end{aligned}
$$

As in the case of the one-point pdf no closed equation for the N-point pdf is obtained. The evolution equation for the N-point pdf contains the N+1-point pdf leading to a hierarchy of equations.

In the dissipation term we can perform a partial integration which eliminates the formally defined operator $L(\mathbf{x}_i,\mathbf{x})$. Additionally, one can recast the pressure and dissipation terms in a way which evidences Galilean invariance:

$$
\begin{aligned}
&\frac{\partial}{\partial t}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t)+\sum_i\mathbf{u}_i\cdot\nabla_{\mathbf{x}_i}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \qquad (28)\\
&=\sum_i\nabla_{\mathbf{u}_i}\cdot\int d\mathbf{u}\int d\mathbf{x}\int d\mathbf{y}\Gamma(\mathbf{x}_i,\mathbf{x}):(\mathbf{u}-\mathbf{u}_i):(\mathbf{u}-\mathbf{u}_i)
\end{aligned}
$$

$$f^{N+1}(\mathbf{u},\mathbf{x},\mathbf{y};\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t)$$
$$-\nu\sum_i \nabla_{\mathbf{u}_i}\cdot\int d\mathbf{u}\int d\mathbf{x}\int d\mathbf{y}\,\delta(\mathbf{x}_i-\mathbf{x})\Delta_{\mathbf{x}}[\mathbf{u}-\mathbf{u}_i]f^{N+1}(\mathbf{u},\mathbf{x},\mathbf{y};\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \qquad .$$

Let us now introduce the notation

$$\frac{\partial}{\partial t}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t)+\sum_i \mathbf{u}_i\cdot\nabla_{\mathbf{x}_i}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \tag{29}$$
$$=-\sum_i \nabla_{\mathbf{u}_i}\cdot\int d\mathbf{u}\int d\mathbf{x}\int d\mathbf{y}A(\mathbf{x}_i-\mathbf{x},\mathbf{u}_i-\mathbf{u})f^{N+1}(\mathbf{u},\mathbf{x},\mathbf{y};\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \qquad .$$

Here, σ denotes the triple $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$. A is an operator which is related to the acceleration and is defined according to:

$$A(\mathbf{x}_i-\mathbf{x},\mathbf{u}_i-\mathbf{u})=-\Gamma(\mathbf{x}_i,\mathbf{x}):(\mathbf{u}-\mathbf{u}_i):(\mathbf{u}-\mathbf{u}_i)+\nu\delta(\mathbf{x}_i-\mathbf{x})\Delta_{\mathbf{x}}[\mathbf{u}-\mathbf{u}_i] \qquad . \tag{30}$$

One may also add a random force field, which is Gaussian as well as δ-correlated in time. Then a diffusion term of the form

$$\frac{1}{2}\sum_i\sum_j \nabla_{\mathbf{u}_i}Q(\mathbf{x}_i,\mathbf{x}_j)\nabla_{\mathbf{u}_j}f^N(\{\sigma_j\};t) \tag{31}$$

has to be included.

It is important to formulate the following invariance properties of the hierarchy. Due to Galilean invariance $f^N(\{\mathbf{u}_i+\mathbf{c},\mathbf{x}_i-\mathbf{c}t,\mathbf{y}_i\};t)$ solves the hierarchy provided that $f^N(\{\mathbf{u}_i,\mathbf{x}_i,\mathbf{y}_i\};t)$ is a solution of the hierarchy. Neglecting the viscous terms, it can be shown that if $f^N(\{\mathbf{u}_i,\mathbf{x}_i,\mathbf{y}_i\};t)$ is a solution, then also $\lambda^{3(\gamma+\delta)}f^N(\{\lambda^\gamma\mathbf{u}_i,\lambda^\delta\mathbf{x}_i,\lambda^\delta\mathbf{y}_i\};\lambda t)$ solves the hierachy for each value of λ, provided that

$$\delta-\gamma=1 \qquad . \tag{32}$$

The scale symmetry for an arbitrary value of δ is obviously broken by the viscous term.

As we have indicated above, the Eulerian N-point pdf can be calculated by the corresponding Lagrangian pdf by integration with respect to the initial locations $\mathbf{y}_i$ of the particles. Integrating each equation of the hierarchy (29) we obtain a corresponding one for the Eulerian probability distribution function. This hierarchy has already been presented by Lundgren [23] as well as Ulinich and Lyubimov [24].

Let us briefly comment on the question why a whole hierarchy of evolution equation arises. The mathematical treatment of fluid flow leads to a field theory. Considering only a finite number of fluid particles, therefore, yields a description with restricted information. This reduction of information shows up in the existence of an unclosed hierarchy of evolution equations for the joint velocity-position pdfs. A closed evolution equation can only be expected to arise when one approaches the continuum description, i.e. when one considers the full probability density functional defined in eq. (18).

IV. EVOLUTION EQUATION FOR THE PROBABILITY DENSITY FUNCTIONAL

This section is devoted to the derivation of a closed evolution equation for the probability density functional $P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t]$. Time differentiation yields:

$$\begin{aligned} &\frac{d}{dt} P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t] = \\ &\quad - \int d\mathbf{y} \{ < \dot{\mathbf{X}}(t, \mathbf{y}) \cdot \frac{\delta}{\delta \mathbf{x}(\mathbf{y})} D[\mathbf{x}(\mathbf{y}) - \mathbf{X}(t, \mathbf{y})] D[\mathbf{u}(\mathbf{y}) - \mathbf{U}(t, \mathbf{y})] > \\ &\quad + < \dot{\mathbf{U}}(t, \mathbf{y}) \cdot \frac{\delta}{\delta \mathbf{u}(\mathbf{y})} D[\mathbf{x}(\mathbf{y}) - \mathbf{X}(t, \mathbf{y})] D[\mathbf{u}(\mathbf{y}) - \mathbf{U}(t, \mathbf{y})] > \} \qquad . \end{aligned} \tag{33}$$

Using the Lagrangian formulation of the Navier-Stokes equation (12) we end up with the following relation:

$$\begin{aligned} &\frac{d}{dt} P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t] + \int d\mathbf{y} \mathbf{u}(\mathbf{y}) \cdot \frac{\delta}{\delta \mathbf{x}(\mathbf{y})} P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t] \\ &\quad = \int d\mathbf{y} \int d\mathbf{y}' \{ \frac{\delta}{\delta \mathbf{u}(\mathbf{y})} \cdot \Gamma(\mathbf{x}(\mathbf{y}), \mathbf{x}(\mathbf{y}')) : \mathbf{u}(\mathbf{y}') : \mathbf{u}(\mathbf{y}') \\ &\quad - \nu \frac{\delta}{\delta \mathbf{u}(\mathbf{y})} \cdot L(\mathbf{x}(\mathbf{y}), \mathbf{x}(\mathbf{y}')) \mathbf{u}(\mathbf{y}') \} P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t] \qquad . \end{aligned} \tag{34}$$

We have arrived at a closed equation determining the evolution of the probability functional $P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t]$. The N-point probability functions $f^N(\{\mathbf{u}_i, \mathbf{x}_i, \mathbf{y}_i\}; t)$ can be obtained from $P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t]$ by functional integration

$$\begin{aligned} &f^N(\{\mathbf{u}_i, \mathbf{x}_i, \mathbf{y}_i\}; t) = \int D\mathbf{u}(\mathbf{y}) D\mathbf{x}(\mathbf{y}) \delta[\mathbf{u}_1 - \mathbf{u}(\mathbf{y}_1)] \delta[\mathbf{x}_1 - \mathbf{x}(\mathbf{y}_1)] \\ &\quad .....\delta[\mathbf{u}_N - \mathbf{u}(\mathbf{y}_N)] \delta[\mathbf{x}_N - \mathbf{x}(\mathbf{y}_N)] P[\mathbf{x}(\mathbf{y}), \mathbf{u}(\mathbf{y}), t] \qquad . \end{aligned} \tag{35}$$

The hierarchy of evolution equations (29) for the N-point pdfs is a projection of the functional equation (34) onto the pdf of N fluid particles according to (35). We mention that, in principle, a projector formalism should be used to pass from the evolution equation for the probability functional (34) to a closed equation for the projected N-point probability distribution.

V. GENERALIZED FOKKER-PLANCK EQUATIONS

If we investigate the problem of turbulence by a hierarchy of evolution equations such as (29) we need to formulate suitable closure schemes. In the following we shall present a formulation of (29) which seems to be more suitable for that purpose. It is influenced by the so-called Lagrangian pdf method [12], [13] , [15] , whose basic idea is to model the acceleration term by a stochastic force. Assuming Markovian properties the model pdf obeys a Fokker-Planck equation of the form

$$\begin{aligned}
&\frac{\partial}{\partial t} f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) + \sum_i \mathbf{u}_i \cdot \nabla_{\mathbf{x}_i} f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \\
&\quad = -\sum_i \nabla_{\mathbf{u}_i} \cdot D^1(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\}) \\
&\quad + \sum_{ij} \nabla_{\mathbf{u}_i} D^2(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\}) \nabla_{\mathbf{u}_j} f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \qquad .
\end{aligned} \tag{36}$$

The question arises, what functional form of the drift term $D^1(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\})$ and the diffusion matrix $D^2(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\})$ has to be chosen in order to obtain an accurate model of turbulent flows. Although the case of a single fluid particle seems to be well-investigated [13], the case of several fluid particles has to be studied in more detail [17], [18], [15]. Interesting models for several fluid particles have been devised by Pumir et al. [19], [20].

Origionally, the random force method dates back to Oboukhov [16], who suggested to use a Fokker-Planck equation without drift term and a constant diffusion term for the single particle case.

In the following we shall derive a generalization of the Fokker-Planck equation directly from the hierarchy (29). To this end we consider equation (29) to be a linear inhomogeneous equation, which can be solved in a straightforward manner (we consider the case of the (N+1)-point pdf):

$$\begin{aligned}
&f^{N+1}(\mathbf{u},\mathbf{x},\mathbf{y};\{\mathbf{u}_i,\mathbf{x}_i,\mathbf{y}_i\};t) = \\
&\quad e^{-(t-t_0)[\mathbf{u}\cdot\nabla_{\mathbf{x}}+\sum_i \mathbf{u}_i\cdot\nabla_{\mathbf{x}_i}]} f^{N+1}(\mathbf{u},\mathbf{x},\mathbf{y};\{\mathbf{u}_i,\mathbf{x}_i,\mathbf{y}_i\};t_0) \\
&\quad - \int_{t_0}^{t} dt' e^{-(t-t')[\mathbf{u}\cdot\nabla_{\mathbf{x}}+\sum_i \mathbf{u}_i\cdot\nabla_{\mathbf{x}_i}]} \int d\sigma'
\end{aligned} \tag{37}$$

$$[\sum_i \mathbf{A}(\mathbf{x}_i-\mathbf{x}',\mathbf{u}_i-\mathbf{u}')\cdot\nabla_{\mathbf{u}_i}+\mathbf{A}(\mathbf{x}-\mathbf{x}',\mathbf{u}-\mathbf{u}')\cdot\nabla_{\mathbf{u}}]$$
$$f^{N+2}(\mathbf{u},\mathbf{x},\mathbf{y};\mathbf{u}',\mathbf{x}',\mathbf{y}';\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t') \qquad .$$

The first term stems from the initial condition.

We remind the reader that we have considered the force free case. Nonrandom forces $\mathbf{F}(\mathbf{x},t)$ can be taken into account by a different evolution operator

$$e^{-(t-t')\mathbf{u}\cdot\nabla_{\mathbf{x}}}\rightarrow Te^{-(t-t')\mathbf{u}\cdot\nabla_{\mathbf{x}}-\int_t^{t'} d\tau\mathbf{F}(\mathbf{x},\tau)\cdot\nabla_{\mathbf{u}}} \qquad . \tag{38}$$

(Here, T denotes Dysons time ordering operator). Whereas the first evolution operator makes the replacement

$$\mathbf{x}\rightarrow\mathbf{x}-\mathbf{u}(t-t') \tag{39}$$

the second evolution operator replaces

$$\mathbf{x}\rightarrow\mathbf{X}(\mathbf{x},t-t') \qquad , \tag{40}$$

where $\mathbf{X}(\mathbf{x},t-t_0)$ is the solution of the set of differential equations

$$\begin{aligned}\frac{d}{dt'}\mathbf{X}(\mathbf{x},t-t') &= \mathbf{U}(\mathbf{x},t-t')\\ \frac{d}{dt'}\mathbf{U}(\mathbf{x},t-t') &= \mathbf{F}(\mathbf{X}(\mathbf{x},t-t'),t')\end{aligned} \tag{41}$$

with the conditions

$$\begin{aligned}\mathbf{X}(\mathbf{x},0) &= \mathbf{x}\\ \mathbf{U}(\mathbf{x},0) &= \mathbf{u} \qquad .\end{aligned} \tag{42}$$

Let us make some remarks on the external force. In three dimensional turbulence the force varies on the so-called integral scale, which is larger than the scales belonging to the inertial scale. That implies that during inertial time scales the relative motion of Lagrangian particles located within the inertial range is not influenced by the external force, i.e. by the mechanism how the turbulence is generated. Therefore, on time scales belonging to the inertial time scale the approximation (39) is sufficiently good for our purposes.

Now we insert the obtained expression into the acceleration term of eq. (29). As a result we arrive at the hierarchy

$$\frac{\partial}{\partial t}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t)+\sum_i\mathbf{u}_i\cdot\nabla_{\mathbf{x}_i}f^N(\{\mathbf{u}_j,\mathbf{x}_j,\mathbf{y}_j\};t) \tag{43}$$

$$
\begin{aligned}
&= -\sum_i \nabla_{\mathbf{u}_i} \cdot \int_{t_0}^{t} dt' [\mathbf{D}^1(\mathbf{x}_i|\{\mathbf{u}_i, \tilde{\mathbf{x}}_i, \mathbf{y}_i\}; t, t') f^N(\{\mathbf{u}_i, \tilde{\mathbf{x}}_i, \mathbf{y}_i\}; t')]_{\tilde{\mathbf{x}}_i = \mathbf{X}_i(\mathbf{x}_i, t-t')} \\
&+ \sum_{ij} \nabla_{\mathbf{u}_i} \cdot \int_{t_0}^{t} dt' [\mathbf{D}^2(\mathbf{x}_i, \tilde{\mathbf{x}}_j|\{\mathbf{u}_i, \tilde{\mathbf{x}}_i, \mathbf{y}_i\}; t, t') \cdot \nabla_{\mathbf{u}_j} f^N(\{\mathbf{u}_i, \tilde{\mathbf{x}}_i, \mathbf{y}_i\}; t')]_{\tilde{\mathbf{x}}_i = \mathbf{X}_i(\mathbf{x}_i, t-t')} \\
&- \sum_i \nabla_{\mathbf{u}_i} \cdot \int d\sigma \mathbf{A}(\mathbf{x}_i - \mathbf{x}, \mathbf{u}_i - \mathbf{u}) \\
&f^{N+1}(\mathbf{u}, \mathbf{X}(\mathbf{x}, t - t_0), \mathbf{y}; \{\mathbf{u}_i, \mathbf{X}_i(\mathbf{x}_i, t - t_0), \mathbf{y}_i\}; t = t_0) \qquad .
\end{aligned}
$$

The last term stems from the initial condition. We want to point out that, in some sense, the present procedure is analogous to a projection operator formalism [26]. It is well-known that the stochastic equations contain initial terms after projection. By some more or less sophisticated arguments, these initial terms are dropped.

Our result (43) takes the form of a generalized Fokker-Planck equation, where the generalized drift term is given by

$$
\begin{aligned}
&D^1(\mathbf{x}_i|\{\mathbf{u}_i, \tilde{\mathbf{x}}_i, \mathbf{y}_i\}; t, t')) = \\
&\quad \int d\sigma \int d\sigma' \mathbf{A}(\mathbf{x}_i - \mathbf{x}, \mathbf{u}_i - \mathbf{u}) \\
&\quad \{[\sum_j \mathbf{A}(\tilde{\mathbf{x}}_j - \mathbf{x}', \mathbf{u}_j - \mathbf{u}') \cdot \nabla_{\mathbf{u}_j} + \mathbf{A}(\tilde{\mathbf{x}} - \mathbf{x}', \mathbf{u} - \mathbf{u}') \cdot \nabla_{\mathbf{u}}] \\
&\quad p^{N+2}(\mathbf{u}', \mathbf{x}', \mathbf{y}'; \mathbf{u}, \tilde{\mathbf{x}}, \mathbf{y}|\{\mathbf{u}_j, \tilde{\mathbf{x}}_j, \mathbf{y}_j\}; t')\}_{\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t-t')} \qquad .
\end{aligned} \tag{44}
$$

The diffusion term takes the form

$$
\begin{aligned}
&D^2(\mathbf{x}_i, \tilde{\mathbf{x}}_j|\{\mathbf{u}_i, \tilde{\mathbf{x}}_i, \mathbf{y}_i\}; t, t') = \\
&\quad \int d\sigma \int d\sigma' \mathbf{A}(\mathbf{x}_i - \mathbf{x}, \mathbf{u}_i - \mathbf{u}) \mathbf{A}(\tilde{\mathbf{x}}_j - \mathbf{x}', \mathbf{u}_j - \mathbf{u}') \\
&\quad \times p^{N+2}(\mathbf{u}', \mathbf{x}', \mathbf{y}'; \mathbf{u}, \tilde{\mathbf{x}}, \mathbf{y}|\{\mathbf{u}_j, \tilde{\mathbf{x}}_j, \mathbf{y}_j\}; t')|_{\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t-t')} \qquad .
\end{aligned}
$$

A formally closed equation has been obtained by the introduction of the conditional probability distribution

$$
f^{N+2}(\sigma'; \sigma; \{\sigma_j\}; t) = p^{N+2}(\sigma'; \sigma|\{\sigma_j\}; t) f^N(\{\sigma_j\}; t) \qquad , \tag{45}
$$

(σ denotes the triple $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$.) A successful description of Lagrangian turbulence statistics can be achieved if this conditional probability distribution can either be approximated or modeled in a suitable way. Thereby, the fundamental symmetries, i.e. Galilean and scale invariance (for $\nu = 0$), have to be retained. Furthermore, incompressibility of the fluid motion should be conserved. This requirement seems to be the major difficulty, since any approximation has a consequence for the pressure term. However, only a correct treatment of the pressure term guarantees incompressibility of the fluid.

VI. SUMMARY

We have formulated a hierachy of evolution equations for the Lagrangian N-point pdf's in close analogy to the one for the Eulerian pdf's presented by Lundgren [23] and Ulinich and Lyubimov [24]. Due to the pressure and dissipative terms the N-point probability distributions couple to (N+1)-point distributions. The existence of a whole hierarchy of evolution equations is due to the fact that a field theory is described by a finite number of points. A closed statistical equation arises when one defines a probability functional. We have formulated an evolution equation for this functional which is the Lagrangian analog of Hopf's functional equation.

Furthermore, we have tried to derive the Lagrangian pdf models [12], which are successful in modeling various aspects of turbulent flows by diffusion processes. Starting from the hierarchy of evolution equations for N-point pdfs we arrived at a generalized Fokker-Planck equation, i.e. a diffusion equation containing memory terms as well as a term steming from the initial condition. The generalized drift and diffusion coefficients are formally expressed by conditional probability distributions, so that the problem is not closed. In a following paper we shall adress the problem of formulating suitable closure approximations [25].

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